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Music in Holland: Consonances According to Simon Stevin

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Abstract. The 17th century was a gold century for Dutch science in general and in particular for the theory of music, still belonging to the physical mathematical disciplines. Beeckman, Stevin and Huygens produced important writings on the subject. In the present paper the conception of consonance for musical intervals of Simon Stevin is presented. A quite strange conception according to most historians of music because Stevin contested the shared opinion for which consonance of two notes occurs for ratios of their pitches expressed by simple integer numbers. For instance, the fifth, one of the most consonant intervals, was unanimously associated to the ratio 3:2 (= 1.5). According to Stevin this was instead a quite crude approximations, the correct value of this interval being $1{:}\sqrt[12]{1/2^7}$ (= 1.498). The present paper does not pronounce about Stevin's approach, it rather wants to discuss one of the proofs appearing in Stevin's musical treatise, *Vande spiegeling der singkonst*. The few historians that commented Stevin's proofs sustained that his reasoning was not so stringent and faulty of paralogism. It will be shown that this not the case; and if Stevin's result was wrong, this depended by experimental errors only.

Keywords: Music theory, Consonances, Acoustics, Simon Stevin, Musical intervals

1. Introduction

The 17th century was a gold century for Dutch science in general and in particular for the theory of music, still belonging to the physical mathematical disciplines. Some skilled mathematicians wrote interesting notes about quantitative musical theories; among them: Simon Stevin (1548-1620), Isaac Beeckman (1588-1637), Dirck Rembrantszoon van Nierop (ca. 1610-1682) and Christian Huygens (1629-1695). But even René Descartes (1596-1650), who merits the inclusion among the Dutch scholars involved in music matters because his long stay in the Netherlands in the first part of the 17th century, is worthy to be considered.

Stevin does not need any introduction as a mathematician, a physicist and an engineer; less known is his role as a scholar of music. Apart from a treatise on music that would be commented later, in the Royal Library in The Hague there are some interesting Stevin's musical sheets, which would testify a not superficial knowledge of music by Stevin (Rasch 1992, p. 188).

Beeckman, a generation younger than Stevin, was possibly more involved than Stevin in music matter. His considerations about music are scattered in his diaries. Descartes is well known for his *Musicae compendium* published posthumously in 1650. He however was scarcely interested in music execution. In an arguably provocative manner, he declared that he was practically deaf in music and could not distinguish a fifth from an octave (Rasch 1992, p. 196). Van Nierop is today the less know of the group. All his scientific works are in Dutch, which partially explains is lack of notoriety today. He wrote in 1659 a musical treatise, the *Wiskonstige Musyka*.

Christian Huygens was the most sophisticated in musical matters of the Dutch mathematicians. He leaned theory and practice of music by his father, Constantijn. Though he published only a brief writing on music in 1691, the *Lettre touchant le cycle harmonique*, very interesting comments can be found in his *Oeuvres*. Notwithstanding his musical skill, only a simple attempt at musical composition has reached us (Rasch 1992, p. 202).

2. The Problem of Consonance

It is an experimental matter of fact that when two notes of different frequencies are played simultaneously by keeping one f_1 fixed and varying the other f_2, starting, for example, from the unison, for certain values of f_2 the ear perceives a harmonious sensation of fusion of the two notes. This sensation, known as consonance, is not perceived by all people in the same way, and perception also varies from culture to culture. The empirical datum, in a mathematical physical theory such as music, is subject to mathematical modeling, which inevitably simplifies its aspects, by canceling 'impurities'.[1]

In the past there have been two fundamental approaches the (neo-)Pythagorean and the Aristoxenian. In the first approach, an arithmetic one, the distance between two notes of frequencies $f_1 < f_2$, produced by a musical instrument for example the monochord, is measured by their ratio $f_2 : f_1$, which is called the musical interval between f_1 and f_2 (a value always greater than unity if one interprets the ratio as a fraction). Two contiguous musical intervals ($f_2 : f_1$) and ($f_3 : f_2$) can be added together giving rise to the interval ($f_3 : f_1$). Thus, with some lexical freedom, interpreting the proportion as a fraction in modern terms, we can say that the sum of two intervals is given by their product. Subtraction between intervals is similarly defined, which instead of multiplication gives rise to division. It is assumed as an a priori assumption that consonance occurs only for intervals represented exactly by simple ratios between integers, in keeping with the Pythagorean tradition that it is the ratios between integers, the only ones possessing implicit purity, that govern the laws of the universe. Among the intervals considered unanimously among the Greeks to be consonant are the octave (2 : 1), the fifth (3 : 2) and the fourth (4 : 3), called by them respectively, diapason, diapente, diatessaron. Other intervals that were considered consonant, especially in the late period, are not considered here because Stevin does not name them.

An alternative approach to the Pythagorean approach is that proposed by Aristoxenus (fl. 350 BCE). According to this approach, a geometrical one, a musical interval is identified with a straight segment; the sum of two contiguous intervals is simply the union of the two segments, the division is their subtraction. In the evaluation of consonances, the ratio of frequencies is not referred to and whole numbers are not considered as particularly meaningful. The uncertainty that exists in determining the precise relationship between frequency values is overcome by identifying a segment to be taken as the unit of measure of the musical intervals: the tone, even better its half, the semitone. Consonances have an integer number of semitones. For example, the resonant interval of fifths is represented by a segment seven semitones long, or equivalently, three tones plus one semitone.

Of course, the different conception of intervals depends on the different ways of measuring the pitches of musical notes. Using modern concepts, it can be said that if they are measured directly by frequencies, the interval must be defined as a ratio; on the other hand, if pitches are measured by the logarithms of frequencies the intervals must be defined as a difference. The two procedures are incompatible if we admit that the same consonances, as seen by Pythagoreans and Aristoxenians, are determined by the same pairs of frequencies. Suppose we measure the frequency ratio for which the consonance occurs and let us assume that it is 3/2 while for a fourth it is 4/3. The difference between the fifth and the fourth is 9/8 (the measure of a tone). Using the Pythagorean approach, we see that the

[1] It should be noted that before the 17th century instead of frequencies, a still confusing concept, people used the lengths of vibrating strings to describe pitches of notes. The description we will make below accordingly should be developed without the use of the concept of frequency. However, for the sake of simplicity and because anachronism does not bring serious consequences, we prefer to use the concept of frequency.

fifth is composed of three tones plus another quantity equal to 256 /243, known as a *limma*, which is less than half a tone: so, the fifth is not three half tones as it must be in Aristoxenus' theory.

The approach of Stevin is intermediate between the two; it is arithmetic in that it defines inter-values by ratios like the Pythagoreans, but these ratios are not between whole numbers; they are in general irrational as is natural for numbers expressing the measure of geometric quantities. This position is justified by the conception of number as expressed in his *Arithmetique*:

> Def. 11. Nombre est celà, par le quel s'explique la quantité de chascune chose (Stevin 1958. vol. 2, p. 495).

If we disregard for the moment the logical difficulties inherent in this definition, it appears very interesting. It is probably dictated by Stevin's practical spirit, whereby if you measure the various quantities of mathematical physics, weights, lengths, times, you never have whole numbers. Nor even rational numbers, at most very close approximations to them. To Stevin, therefore, it must have seemed strange that the interval of consonance, representative of a physical phenomenon, could have such an 'unnatural' value as that postulated by the Pythagoreans.

3. Vande spiegeling der singkonst

Stevin's contribution to music is mainly contained in a short treatise, the *Vande spiegeling der singkonst*, composed in order to teach music to prince Mauritius. It was planned as a part of the *Hypomenta mathematica*, Stevin's masterpiece. However, it was never published and remained in manuscript, which vanished soon after his death in 1620. The manuscript was rediscovered and published for the first time in 1884 (De Haan 1884).

The manuscript consists of two quite different versions, the earlier of which (version 1) was translated into English and appeared in Stevin's *Mathematical works* (Stevin 1955-1966. Vol. 5, pp. 415-466). The other version (version 2) is left in Dutch in (De Haan 1884, pp. 1-47). In the following we will consider version 1 as version 2 has largely been commented in Cohen (1987; 1984, pp. 45-74). The manuscript was sent to the organist Abraham Verheyen who, in a long letter, raised strong criticism (De Haan 1884, pp. 87-91). This would be, according to Cohen (1884, note 46, p. 265), the reason for which Stevin never published his work.

Though little is known of Stevin's musical training, from his quotations it may be deduced that his main reference was Giosefo Zarlino, whose treatise contains, among other things, relevant references to the ancient Greek music. One more reference was Andreas Papius (1552-1581), for his discussion of the consonances, in particular of the fourth. The *Vande spiegeling* is organized *more geometrico* with definitions (6), postulates (2) and theorems (1). The first definition introduced the concept of a *step*, as the interval between two consecutive notes of a musical scale:

> 1*st Definition*. Step is the next subsequent ascent which one rises in natural singing, of which the smaller variety is called minor step, the larger, major step (Stevin 1955b, vol. 5, p. 423).

The second definition concerns the musical scale he considers natural for singing; it is defined by the following sequence of major (t) and minor (s) steps: t-t-s-t-t-t, which corresponds to the diatonic genus, or, in modern terms, to a C-major scale. In fact, there is no objective reason why the one Stevin chose is a more natural scale, than either the enharmonic or the chromatic scale, other than the fact that it was the most widely used scale in his day.

Two definitions (5 and 6) follow that give a name to the intervals in an octave. First, the major and minor step are redefined as a whole tone and a semitone respectively, then the intervals between a given note and a reference note are defined in the usual way. For instance, a fifth is the interval given by three whole tones and one semitone.

The definitions are followed by two postulates. The first postulate has a physical connotation, and its discussion is the only part of Stevin's treatise in which considerations about the nature of sound are made.

> Postulate. We postulate that as one part of a string is to another, so is the coarseness of the sound of the one to that of the other (Stevin 1955b. Vol. 5, p. 425).

Note that the postulate's reference is not to singing, as done so far, but to the sound emitted by a vibrating string. In substance, by using modern terms and concepts, the postulate says that the pitch of a note depends on the length of the string that produces it. Actually, in Stevin's time there was no clear understanding of the concept of frequency of vibration. Stevin spoke of *coarseness* (lower pitches) and *finesses* (higher pitches), the former being greater for longer strings.

The second postulate states that all tones and semitones in a natural scale are equal.

> 2*nd Postulate*. All whole tones to be equal and likewise all semitones to be equal (Stevin 1955b. Vol. 5, p. 427).

The postulate is followed by the following table (Tab. 1) showing the value and name of the intervals contained in an octave.[2]

Table 1

	Theorem. 2)			*Proposition*
	1		Selftone	First (prime)
	$\sqrt{\ }^{(12)}\ 1/2$		Semitone	Minor second
	$\sqrt{\ }^{(6)}\ 1/2$		Whole tone	Major second
	$\sqrt{\ }^{(4)}\ 1/2$		One-tone-and-half	Minor third
	$\sqrt{\ }^{(3)}\ 1/2$		Ditone	Major third
	$\sqrt{\ }^{(12)}\ 1/32$		Two-tone-and-half	Fourth
As 1 is to	$\sqrt{\ }\ 1/2$	So one tone to the other	Tritone	Bad major fourth or bad minor fifth
	$\sqrt{\ }^{(12)}\ 1/128$		Three-tone-and-half	Fifth
	$\sqrt{\ }^{(3)}\ 1/4$		Four-tone	Minor sixth
	$\sqrt{\ }^{(4)}\ 1/8$		Four-tone-and-half	Major sixth
	$\sqrt{\ }^{(6)}\ 1/32$		Five-tone	Minor seventh
	$\sqrt{\ }^{(12)}\ 1/2048$		Five-tone-and-half	Major seventh
	$1/2$		Six-tone	Double-first (octave)

The table restates simply the postulate; at least this would be the impression of a modern reader. Indeed, the postulate implies that the intervals inside a musical scale makes a geometrical progression. By assuming that an interval of octave is divided into 12 semitones, as usual in music theory, the

[2] The way Stevin indicates intervals that is, $(1{:}\sqrt[12]{1/2^n})$, may seem strange to a modern reader who would have written, $(\sqrt[12]{2^n}{:}1)$, representing the same numerical value. But in Stevin's time, when intervals were measured by string lengths instead of frequencies, the octave was generally denoted by the ratio 1 : 2, a ratio less than 1, instead of 2 : 1, a ratio greater than 1. Stevin makes a compromise, denoting the octave by ½, but defining the interval by a ratio greater than 1.

common ratio of the geometrical progression is given by $(1:\sqrt[12]{1/2})$ which is also the ratio of a semitone. A whole tone is given by the ratio $(1 : \sqrt[6]{1/2}) \approx (1{:}\ 0.891)$, a little bit lesser of the tome of the Pythagorean scale: $(8 : 9) \approx (1{:}\ 0.889)$. The interval of fifth, defined by seven semitones is $(1:\sqrt[12]{1/128}) \approx (1{:}\ 0.6674)$, somewhat larger than $(3 : 2) \approx (1{:}\ 0.6667)$.

Stevin knew for sure what a geometrical progression was, but he probably could not think very simple to find its common ratio. The table of intervals (see above) is labeled as *vertooch*, that the editor of the *Vande spiegeling* translates as *theorem*. This means that probably its deduction from the second postulate was not as trivial for Stevin as a modern may assume. For the modern reader, Stevin could have found the tone by simply performing two square and one cubic root, this was in his possibility. But he did not proceed in such a way. His starting point seems to be that a fifth has a ratio $(1:\sqrt[12]{1/128})$. By subtracting it from an octave (2 : 1) one obtains an interval of a fourth, two tones and half, $(1:\sqrt[12]{1/32})$, a well-known result for a musician, which subtracted from the fifth gives a as $(1 : \sqrt[6]{1/2})$, from which the half tone $(1 : \sqrt[12]{1/2})$ is obtained.

The text that follows the statement of the postulate is qualified by Stevin as a proof; it is divided into three sections with the following titles:

1. On ratio in general.
2. Comparison of geometrical ratio with musical ratio.
3. On the ratios of singable sounds according to the opinion of the Greeks.
4. Of the true ratios of natural tones.

In the first section Stevin criticized the concept of proportionality of Greeks, who accepted three different kinds of proportion: arithmetic, geometric, and harmonic. He maintained that this multiplicity was due to the nature of the Greek language (as well as most of modern languages), because it does not allow for a clear idea of the concept of proportion. The Dutch language is more precise because its words are composed in such a way as to make it easy to understand their meaning. And *proportion* is rendered as equal ratio (*everedenheyt*), which makes it possible to exclude arithmetic and geometric proportions from the list of proportions, leaving only geometric proportions.

In the third section Stevin stated that the choice of the value for the intervals of consonances according to the Greeks is arbitrary and approximate. For instance, the fifth is only approximated by the ratio (3 : 2); its correct value is instead $(1{:}\ \sqrt[12]{1/128}) = (1.498{:}1)$. The error is due to the difficulty of the human ear in appreciating the exact ratio: "And although the Ancients perceived this fact, nevertheless they took this division to be correct and perfect, and preferred to think that the defect was in our singing, (as if one should say: the sun may lie, but the clock cannot)" (Stevin 1955b, vol. 5, pp. 431-433).

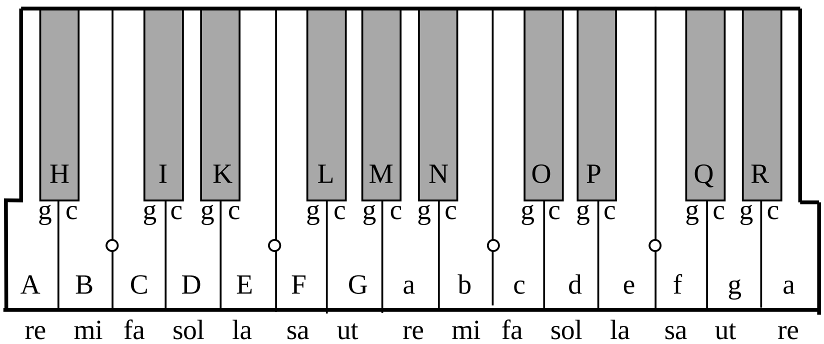

Fig. 1. Stevin's keyboard

The last section tried to prove the postulate 2 by performing a simple experiment. It consists in tuning a keyboard, a harpsichord or an organ for instance. With reference to the keyboard in Fig. 1, assume to choose the key F as the starting note in the tuning procedure. Then tune one of the keys H_i (A, B, C, ….) $i = 1, 2, …, 11$, H_1 for instance, such that H_1F makes an interval of a fifth. From H_1 find another key H_2 such that H_1 H_2 makes a fifth. Continue in this way for 11 times to reach the key H_{11}, pay attention that if a key would be maintained inside the octave by raising or lowering this interval by one octave.

After the 11 cycles, Stevin carried out his experiment by playing the two keys F and H_{11}. According to him, he obtained an interval of fifth:

> This being so, experience shows that Hand F make a perfect fifth, and although this is considered a common and certain rule by all those who are skilled in this matter, yet to convince those who should doubt it 1 thought fit to use the authority of ... (Stevin 1955b, vol. 5, p. 437)

With this result, by using reduction to the absurd, Stevin can prove that all the semitones, and consequently the tones, are equal.

Before commenting on his procedure, some incongruences should be evidenced. First, Stevin stated, here and there in his treatise, that the true measure of a fifth is not 3: 2 but 1: $\sqrt[12]{1/128}$. One could imagine that starting from this assumption, a postulate indeed, he could prove the equality of the semitones. Actually, he never used this assumption, but rather proved it. Secondly, it must be said it is not true that all who are skilled in music would accept that a cycle of 12 fifths closes. Indeed, by assuming for a fifth the generally accepted value of (3 : 2), 12 cycles of fifths, starting from F = 1, gives a ratio of 1.0136, greater than 1. This means that playing H_{11} and F, an interval shorter than a fifth is obtained; this interval is called *wolf fifth* and is highly dissonant. The difference between a fifth and a *wolf fifth* is known as a *Pythagorean comma*.

Let now come back to the experiment. There is a problem here. Indeed, Stevin did not specify how he tuned his keyboard in practice, that is how he recognized fifths and octaves. There is no evidence in the musical writings by Stevin that his ear could be so good to allow him tuning simply by ear; but he could have had help from an experienced musician.

4. Criticisms to Stevin's Approach

Stevin's position at first glance seems reasonable. Indeed, there is no reason why consonant intervals should be given by simple ratios between integers. The thesis that an octave is made up of 12 equal semitones is fascinating, and is the same result obtained by Aristoxenus' geometric approach. But equally it cannot be proved at the rational level.

The objections that a scholar just after Stevin, or even contemporary, could make to his position were many:

1. Stevin contradictorily argued that experience, because of the imperfection of the human ear, cannot distinguish, for example, whether a fifth is defined by the ratio (3:2); however, then he believed that with the experience of tuning the organ it could be shown that the fifth is indeed (1: $\sqrt[12]{1/128}$).
2. Basically, the totality of musicians considered an interval expressed by the proportion between whole numbers to be more consonant. In this case, the thesis of subjectivity of sensations fails because there is a concordance among multiple listeners. One could, however, argue the thesis of cultural dependence, that is, musicians are so used to being told that consonance is 3:2 that they come to believe it even if it is not true.

3. Already in Stevin's time, or shortly thereafter, there were objective criteria, theoretical or experimental, for judging a consonance and they all leaned in favor of the Pythagorean thesis. Benedetti, whom Stevin most likely did not know, had developed a theory of consonance based on concordance, the same as Galileo's and Beeckman's, and the theory of concordance explains consonance by ratio of simple integer number, and thus rejects Stevin's thesis. While it is true that the theory of concordance is only a theory, it is very convincing by having a mechanistic basis. There was also an experimental method that could be pursued without any musical skill in tuning musical instruments, which is therefore independent of the sensitivity of the human ear. The method is based on the phenomenon of beats. When two notes whose frequencies are close to a consonance are played, a phenomenon occurs whereby the amplitude of the sound increases and decreases periodically; this phenomenon is known as beats. In the absence of beats, a better consonance is noticed than when there are beats. Experience conducted, for example, with the organ, whose pipe emits sounds that last long enough, shows that the absence of beats occurs when the frequency of the two notes is in simple ratios.

One could certainly rebut all these objections, but this would be meaningless since there are no equally stringent arguments to justify Stevin's thesis. If he had an oscilloscope fed with two sinusoidal signals, he would have seen that something extraordinary happens only when the frequencies of the two signals stand in simple ratios. Thus, the seemingly implausible Pythagorean thesis that physical phenomena are describable through of integers, in this case works.

5. Conclusions

The *Vande spiegeling* is certainly an interesting work and could have provided much stimulation for quantitative music theorists at a time when traditional intonation was entering a crisis. But it did not. Stevin's work was in fact read by few, as far as it is known perhaps only by Beeckman and Huygens, partly because it was written in Dutch. And those few who did read it gave an unflattering assessment. Beeckman in a letter to Mersenne October 1, 1629, wrote that he had initially embraced Stevin's ideas and then rejected them because contrary to his theory of concordance (Beeckman (1939-1953), vol. 4, p. 157). Huygens, who was the last person to read *Vande spiegeling* before it disappeared, expressed harsh criticism of the theory of irrational consonant intervals: "And those who dared [...] that the 5 does not consist of the ratio 3 : 2, either do not have an ear able to judging or they believe to have a good reason for that; but they conclude wrongly" (Huygens 1888-1950, vol. 20, p. 32).

Thus Stevin in fact made no substantial contribution to the history of music at least until 1884, when his work was published by De Haan. Since then, some historians have wanted to see Stevin as a forerunner of uniform temperament and the first to have provided accurate value for the intervals that define it. This judgment, as extensively documented by Cohen (1987) is fundamentally incorrect since Stevin's problem was that of the exact value of consonance and not its best approximation, which was the purpose of a temperament. One also wants to see an influence of Stevin on Schoenberg's dodecaphonic music, but this is maybe too a forced judgment (Devreese & Berghe 2008, p. 257).

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