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# Exchanges in the nucleus in the 1930s: Werner Heisenberg and Ettore Majorana

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*Abstract*: The discovery of the neutron in 1932 provided the key picklock that allowed quantum mechanics to be applied to the problem of nuclear structure. The first, moderately satisfactory attempt was undertaken by Heisenberg, who had already pioneered the introduction of exchange interaction in the quantum mechanics of identical atoms and had first applied them to the helium atom. While Heisenberg took inspiration from a different incarnation of exchange interactions, introduced in molecular physics to explain covalent bonds, Majorana went back to the original idea and proposed (at first independently of Heisenberg) his own version, which overtook several drawbacks of Heisenberg's theory and gave results closer to experiments. In this contribution, we reconstruct this story, including a detailed discussion of Majorana's unpublished notes on this subject, which in fact contain much more material than what was published.

*Keywords:* Quantum Mechanics, Exchange Interactions, Nuclear Theory

## 1. Introduction

The prehistory of the concept of exchange interaction can be traced back to 1926, when W. Heisenberg introduced the idea of quantum resonance for the quantum mechanical description of a system of identical particles (Heisenberg, 1926). Then he applied it to the spectrum of the helium atom, managing for the first time to give the correct explanation. In 1928, the same idea was crucial for his successful explanation of the strong spin-spin interaction at the basis of (anti-)ferromagnetism (Heisenberg, 1928). In 1927, W. Heitler and F. London brought the idea of exchange interaction into the domain of molecular physics and made a seminal contribution, leading to the birth of the quantum theory of the homopolar chemical bond (Heitler & London, 1927). Here a notion of electrons exchanging their places around the two different nuclei was introduced (which is valid also for the single electron case, e.g. the $H_2^+$ ion), termed *Austausch*, which is an evolution of Heisenberg's exchange between two states with different energy levels (Carson, 1996a).

The foundations were laid in view of the subsequent step, the application of quantum mechanics to the explanation of nuclear structure, which took place in 1932 after the discovery of the neutron (Chadwick, 1932). Indeed, the existing picture of the nucleus, as a composite of protons and electrons, was a source of many inconsistencies; among them it is worth to mention the wrong statistics of the $^{14}$N nucleus and the violation of the momentum-position uncertainty relation for the confined electrons. Furthermore, the experimental evidence of the continuous nature of the $\beta$-ray spectrum, and mainly its striking contradiction with the energy and momentum conservation principle, was still awaiting a theoretical explanation. All these critical issues led many scientists, like N. Bohr, to deny the validity of quantum mechanics at nuclear scales.

Quantum mechanics entered the nuclear domain thanks to Heisenberg, who built up the first theory of nuclear structure by describing the interaction between protons and neutrons via an exchange mechanism (Heisenberg, 1922a; 1932b; 1933). Almost in the same period E. Majorana put forward his own theory of nuclear forces (Majorana, 1933a; 1933b), which improved Heisenberg's one while solving its main drawbacks. As a further bonus, a better agreement with experimental results was obtained. The aim of this contribution is to retrace this story by focusing on the main ideas as well as the differences between Heisenberg's and Majorana's approach to nuclear force theory (Section 2 and 3, respectively). A detailed analysis of Majorana's unpublished research notes (Esposito *et al.*, 2008) is also provided (Section 4), followed by our conclusions and perspectives.

## 2. Heisenberg theory of nuclear structure

The discovery of the neutron (Chadwick, 1932) triggered the first attempts to bring quantum mechanics into the nuclear domain. This challenging task was pursued by Heisenberg, who tried to overcome the problems related to the picture of a nucleus built of protons and electrons by assuming that the neutron was a nuclear constituent, together with the proton. In this way all the critical issues could be swept inside the neutron:

> The neutron will be taken as an independent fundamental particle which, however, can split, under favourable conditions, into a proton and an electron, violating the law of conservation of energy and momentum. (Heisenberg, 1932a, p. 1; translated in Brink, 1965, p. 145)

Heisenberg kept using this hybrid point of view throughout his celebrated papers on nuclear theory, written between June and December 1932 (Heisenberg, 1932a; 1932b; 1933). Furthermore, in his third paper, he stated that the two points of view were to be considered as "incompatible types of properties in quantum mechanics" (in the sense of the complementarity principle):

> The assumption that the neutron appears as an elementary particle when considering spin and statistics, while it resembles a composite structure when considering polarizability, decay, etc. leads to the problem of merging two incompatible types of properties in quantum mechanics. (Heisenberg, 1933, p. 595; translated in Esposito, 2015, p. 34)

Heisenberg modelled the $p - n$ interaction in analogy with that between the hydrogen atom and the hydrogen ion in the molecular ion $H_2^+$, so that it could result very strong within the nuclear region while satisfying a saturation property. As such it appears to be due to the exchange of a spinless Bose electron between the neutron and the proton:

> If one brings a neutron and a proton to within a distance comparable to the dimensions of the nucleus, then – in analogy with the $H_2^+$ ion – a change of place of the negative charge will occur with a frequency given by a function $\frac{1}{h}J(r)$ of the distance between the two particles. The quantity $J(r)$ corresponds to the Austausch - or more correctly the Platzwechsel - integral of molecular theory. One can illustrate this change of place again with the picture of electrons that have no spin and obey Bose statistics. But it is probably more correct to regard the exchange integral $J(r)$ as a fundamental property of the proton-neutron pair, without wanting to reduce it to motion of electrons. (Heisenberg, 1932a, p. 2; translated in Carson, 1996b, p. 104)

Notice the careful wording in the above passage. Indeed, Heisenberg's $p - n$ interaction can be viewed as analogous either to that occurring in the hydrogen molecular ion (*Platzwechsel*), or to that occurring in the helium atom (*Austausch*) (Carson, 1996b; Miller, 1994). Indeed, if we forget about the mediating electron, we can view the effect as due to proton-neutron exchange (again an *Austausch*). Starting from this consideration, Heisenberg introduced the notion of isospin, which in the subsequent would have gained a growing consensus within nuclear physics. Within the new picture the neutron and the proton were identified with the two different states of the nucleon, so that Heisenberg's Hamiltonian reads (Heisenberg, 1932a):

$$H = \frac{1}{2M}\sum_k \boldsymbol{p}_k^2 - \frac{1}{2}\sum_{k<l} J(r_{kl})(\rho_k^\xi \rho_l^\xi + \rho_k^\eta \rho_l^\eta) + \frac{1}{4}\sum_{k>l} K(r_{kl})(1+\rho_k^\zeta)(1+\rho_l^\zeta) + \\ + \frac{1}{4}\sum_{k>l} \frac{e^2}{r_{kl}}(1-\rho_k^\zeta)(1-\rho_l^\zeta) - \frac{1}{2}D\sum_k (1+\rho_k^\zeta) \tag{2.1}$$

where $\rho^\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\rho^\eta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\rho^\zeta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, with $\rho^\zeta = +1$ for the neutron and $\rho^\zeta = -1$ for the proton, respectively.

The exchange integral $J(r)$ is unspecified, apart from its sign and the requirement that it produces a short-range interaction. An additional, weaker exchange interaction $K(r)$ is assumed between neutrons, while only the electrostatic repulsion acts between protons. A mass defect term $D$ of the proton with respect to the neutron is also considered. Furthermore, $M$ is the proton or neutron mass, $r_{kl} = |\boldsymbol{r}_k - \boldsymbol{r}_l|$ are distances of the order of the nuclear dimensions and $\boldsymbol{p}_k$ is the momentum of the $k$-th nucleon. The matrix in front of $J(r)$ can be rewritten as $\rho_+^k \rho_-^l + \rho_+^l \rho_-^k$, whose effect is to take a state with a proton in position $\boldsymbol{r}_k$ with spin $\boldsymbol{s}_k$ and a neutron in position $\boldsymbol{r}_l$ with spin $\boldsymbol{s}_l$ and turn it into state with a neutron in position $\boldsymbol{r}_k$ with spin $\boldsymbol{s}_k$ and a proton in position position $\boldsymbol{r}_l$ with spin $\boldsymbol{s}_l$.

Heisenberg's theory contained many insights, which paved the way to a successful description of nuclear interactions, but gave rise to a number of drawbacks, demanding further improvements. In fact, the symmetry properties of the ground state wave function of the deuteron resulting from the choice of the sign of $J(r)$ led to the saturation of the deuteron, in contrast with experimental data pointing to the $\alpha$-particle. Another critical point was the collapse of the nucleus, arising from the extension of the Thomas-Fermi method to nuclei (Heisenberg, 1932a), which required the imposition of a cut-off to potential energy at short distances. These improvements were carried out by Majorana, whose pivotal contributions (published as well as unpublished) are the subject of the subsequent Sections.

## 3. Majorana's theory of nculear structure: the published side

Around 1930, the interest of E. Fermi's group in Rome gradually shifted from atomic and molecular physics to the new frontier of nuclear physics. Majorana was the first to work on nuclear physics in Rome, starting right after the publication of G. Gamow's pioneering paper on alpha decay (Esposito *et al.*, 2008). His calculations became part of the thesis "The mechanics of radioactive nuclei", which he defended in July 1929. Then he continued studying nuclear physics without publishing anything. In 1929-30, he focused mainly on nuclear reactions induced by alpha particles, developing a full, consistent quantum-mechanical theory of quasi-stationary states (Di Grezia & Esposito, 2008).

As recalled by his colleagues, mainly E. Amaldi (Amaldi, 1968; Recami, 2011), after reading some notes published by Joliot & Curie in 1932, Majorana immediately interpreted their results as due to a neutral massive particle, i.e. a neutron, and before the Easter of that year (before the appearance of Heisenberg's papers) he sketched a theory of nuclear structure based on its presence as a nuclear constituent. However, he stubbornly refused to publish his findings, until March of the following year (1933), when he was in Leipzig, encouraged by Heisenberg himself. Then he published two slightly different versions of his work (Majorana, 1933a; 1933b; Bassani, 2006).

As already pointed out, Heisenberg's theory finally allowed a quantum-mechanical description of nuclear structure, while giving some predictions that, unfortunately, were contradicted by experiments.

In particular, the deuton rather than the alpha particle would be a closed shell; furthermore, its spin would be 0 instead of 1 (this fact would have been known only later, in 1934). Finally, it predicted a very large binding energy for heavy nuclei, which Heisenberg circumvented by including by hand a repulsive short-range interaction. Thus, Majorana started his paper by a lucid critical analysis of Heisenberg's assumptions and results:

> In the absence of other guiding criteria, Heisenberg was guided by an analogy presumably existing between the normal neutral hydrogen atom and the neutron, the last one being supposed – as commonly accepted – to be composed by a proton and an electron... The use of such an analogy is difficult to justify since, if the neutron would be effectively composed by a proton and an electron, their binding could not be described by the present theories, which would lead to associate the Bose-Einstein statistics and an integer multiple of $\frac{h}{2\pi}$ for the mechanical moment to the neutron, contrary to fundamental assumptions. These come directly from empirical properties of the nuclei, and we cannot give them up. Given the present state of our knowledge, it is thus preferable to try to obtain the law of interaction between the elementary particles being guided by simplicity only, in order to predict the most general and characteristic properties of nuclei... Our problem is, then, to find the simplest law of interaction between any elementary particle – protons and neutrons – leading to the definition of an impenetrable matter as long as Coulomb repulsion may be neglected. (Majorana, 1933a, pp. 559-560; translated in Esposito, 2015, p. 35)

The impenetrability property of nuclear matter, i.e. the fact that nuclear matter has a constant density, so that the volume of a nucleus and its binding energy are proportional to the number of nucleons, is crucial for Majorana, who sought to account for it in the simplest possible way. A viable solution could be to take an attractive force for large distances and a short-range strong repulsive force in order to model the impenetrability of particles, but Majorana didn't pursue further this idea and followed a different strategy:

> We shall, therefore, try to find another solution and introduce as few arbitrary elements as possible. The main problem is this: How can we obtain a density independent of the nuclear mass without obstructing the free movement of the particles by an artificial impenetrability? We must try to find an interaction whose average energy per particle never exceeds a certain limit however great the density. This might occur through a sort of saturation phenomenon more or less analogous to valence saturation. (Bassani, 2006, p. 188)

As such, by resorting to Dirac's formalism (Dirac, 1930) he wrote down the following exchange interaction:

$$(Q', q' \mid J \mid Q'', q'') = -\delta(q' - Q'')\delta(q'' - Q')J(r) \tag{3.1}$$

where $r = \mid q' - Q' \mid$, $Q$ and $q$ being the position coordinates of a neutron and a proton. At odds with Heisenberg's exchange interaction, Majorana's one had an opposite sign and exchanged only the position coordinates of proton and neutron without affecting their spins. It didn't rely on the isospin formalism and, interestingly, there is no reference to the exchange of electrons. Concerning $J(r)$, it was assumed to be positive, and its functional form would have been discussed only at the end of the paper. Majorana's choice allowed him to obtain an exchange force leading to the saturation of the $\alpha$-particle, in agreement with experiments:

> Thus we find that both neutrons act on each proton in the $\alpha$-particle instead of only one and viceversa, since we assume a symmetrical function in the position coordinates of all protons and neutrons (which is true only if we neglect the Coulomb energy of the protons). In the $\alpha$-particle all four particles are in the same state so that it is a closed shell. If we proceed from an $\alpha$-particle to heavier nuclei we can have no more particles in the same state because of the Pauli principle. Also, the exchange energy is usually large only if a proton and a neutron are in the same state and we may expect, which agrees with experiments, that in heavy nuclei the mass defect per particle is not noticeably bigger than in the $\alpha$-particle. (Bassani, 2006, pp. 189)

In his Solvay lectures of 1933, Heisenberg acknowledged the superiority of Majorana's theory over his own theory. He also graphically compared his expression of the exchange interaction with Majorana's, and expressed the latter in terms of the isospin formalism as (Heisenberg, 1934):

$$\frac{1}{4}J(r_{kl})(\rho_k^\xi \rho_l^\xi + \rho_k^\eta \rho_l^\eta)(1 + \sigma_k \sigma_l) \tag{3.2}$$

After stating at length his hypotheses, Majorana reported computations performed in the case of a very large number of nucleons (which may be considered to be the case of nuclear matter). He showed that the energy per nucleon has a minimum in correspondence of a value of the nucleon density independent of the mass of the nucleus. Thus, the saturation in the energy is indeed obtained without postulating a short-range strong repulsive interaction, independently of the functional form of $J(r)$.

He concluded the paper by briefly considering some functional forms for the interaction, including what is now known as Yukawa's potential (Yukawa, 1935). Then he commented on some results obtained with these potentials, without going into details. At first, he proposed the expression $J(r) = \lambda\frac{e^2}{r}$, involving a single arbitrary constant, but soon he discarded it because it is divergent and "seems to provide too small a ratio for the mass defects of the $\alpha$-particle and the hydrogen isotope" (Bassani, 2006, p. 193). Thus, he concluded that an expression with at least two arbitrary constants was needed, such as $J(r) = Ae^{-br}$ but gave up pursuing this alternative as well, since "it has been shown that the first statistical approximation can lead to considerable errors however large the number of particles" (Bassani, 2006, p. 193). Indeed, for large nuclei, the proton-proton Coulomb repulsion would make the nucleon density not constant.

## 4. Majorana's theory of nuclear structure: the unpublished side

In his Quaderni, Majorana carried out a lot of calculations in nuclear physics. The most interesting ones for our purposes are contained in Quaderno 17 (started on 20 June 1932), where calculations concerning nuclear potentials were made (Esposito *et al.*, 2008, pp. 340-368). These calculations are both a preliminary study for the paper that would be published the following year and an extension of it. In particular, he performed detailed computations involving both the potentials forms mentioned at the end of his 1933 paper (Majorana, 1933a; 1933b). The relevant sections of Majorana's notes contain the following material:

1. Definition of the interaction and of nucleon wave functions.
2. Calculation of an integral relevant for the potential energy.
3. Calculation of the nucleon density.
4. Interaction potential $J(r) = \lambda\frac{e^2}{r}$, statistical computations.
5. Interaction potential $J(r) = Ae^{-br}$, statistical computations.
6. Light nuclei with the interaction potential $J(r) = \lambda\frac{e^2}{r}$.
7. Light nuclei with interaction potential $J(r) = Ae^{-br}$.

More in detail, Majorana started by considering a generic nucleus, writing down the wave functions of neutrons and protons as Slater determinants and defining the exchange interaction as in the paper. He also considered, like Heisenberg in his first paper (Heisenberg, 1932a), the interaction of two nuclei. Then he proceeded by carrying out elaborate computations concerning the applications to nuclei and nuclear matter. The starting point was the calculation of the relevant integral:

$$\int_{q<R}\int_{q'<R'} \frac{dq\, dq'}{\mid q - q' \mid^2} \tag{4.1}$$

In both cases $R < R'$ and $R > R'$ (these distances are related to proton and neutron densities). This integral is proportional to the potential energy density of a system of nucleons, a result which is subsequently used to compute the nucleon density by using the virial theorem $T = \frac{1}{2}V$ (after an attempt to find it from minimization of the total energy).

Focusing on the first of the two explicit potentials, Majorana derived an expression for the energy (associated with the exchange interaction) per nucleon in nuclear matter, under the hypothesis that the minimum kinetic energy states are occupied. The energy obtained was then evaluated for a large nucleus by means of a statistical method. A numerical estimate was provided in the case $\lambda \sim 1$.

Similar steps were performed with the second explicit potential (in the form $J(r) = A\, e^{-\frac{r}{\epsilon}}$). After very elaborate computations, Majorana obtained a series of numerical tables for the average kinetic and potential energies in various cases. In both cases, he started by taking into account electrostatic repulsion, but in the end he considered a 'zeroth approximation' in which this contribution was neglected, under the assumption of constant densities.

In the last part, Majorana considered light nuclei for both potentials. In the first case he wrote down the Schrödinger equation for the alpha particle and made a rough estimate of the energy. However, the analogous calculation for the deuteron seems to be missing. Concerning the second potential he focused on the deuteron, then he wrote the Schrödinger equation and computed the expectation value of the Hamiltonian for a trial wave function, probably using a variational principle to estimate the ground state energy. Again, the analogous computation for the alpha particle is missing, but the section ends with a short discussion of the wave function of two alpha particles, in connection with their statistics.

## 5. Conclusions

In this contribution, we reconstructed the pivotal role played by Heisenberg and Majorana in the development of the theory of nuclear forces.

Heisenberg's theory, despite its shortcomings, represents the first instance of quantum mechanical description of atomic nuclei. Heisenberg described proton-neutron interactions as quantum exchange (or resonance) forces in two different ways, similar to those taking place among electrons in the helium atom and in the ionized hydrogen molecule, respectively.

Majorana's theory most likely predated Heisenberg's one, and at the same time resolved its main shortcomings, as Heisenberg himself admitted. Like all of Majorana's work, this one is backed up by pages and pages of calculations, not all results of which found their way into the published paper. However, not all computations relevant to the published paper seem to be there either.

Heisenberg's theory was nevertheless a very important step forward, and it contained elements of truth, namely the idea of the interaction mediated by a particle (not an electron, though) and the presence of a repulsive core.

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