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Joseph Sauveur and the fixed sound

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Abstract: At the turn of the 18th century, the development of musical instruments, the increase in the size of orchestras, and the diffusion of music on an European scale led to a growing need for standardizing the frequencies associated with notes. This was a challenging undertaking, as there were no instruments capable of measuring sound frequencies, and standardization was largely based on auditory perception. In the early 18th century, Jospeh Sauveur, the leading acoustician of the time, put forth two methods for achieving this goal. The initial approach was based on the theory of beats, which he was just beginning to comprehend. The objective of this method was to generate a sound with a frequency of 100 Hz, the fixed sound, using an organ pipe. A second method was to identify the equation that describes the relationship between the frequency of a vibrating string and its characteristic parameters. This law was known since the end of the 16th century, but only in a relative sense. That is, it was known that the frequency was proportional to certain parameters, but the constant of proportionality was not known, and it was not straightforward to determine this quantity experimentally. Sauveur proposed an analytical method that led to an equation that was within 1% of the value now considered correct. The following paper analyses Sauveur's two approaches, using unpublished writings.

Keywords: Acoustics, Absolute Frequency, Joseph Sauveur, Theory of Music

1. Introduction

Joseph Sauveur (1653-1716), a French mathematician and member of the Académie des Sciences of Paris since 1696, is regarded as the founder of modern acoustics. He presented his ideas on the subject in five memoirs, published between 1701 and 1713. In addition to addressing fundamental theoretical music issues such as the division of the octave and the explanation of consonances and higher harmonics, he considered a pressing contemporary problem: the absolute measurement of frequencies, i.e. their value in Hz. It was now considered indispensable for the management of orchestras that had increasingly richer staff and were taking on an international character that imposed a problem of standardizing sounds, which was impossible without being able to measure their frequency.

By the late 1600s, the idea of the periodic nature of sound was well established. The terminology, however, was still uncertain; for example, there was no common use of the term frequency, but only locutions such as speed of vibration, oscillations in a fixed time, etc. The name frequency appeared but was not emphasized in some of Gassendi's writings (Baskevitch, 2005, pp. 25-28). Aside from the nomenclature, what was missing was the acknowledgment of the harmonic nature of sound. Sauveur, like Galileo and Mersenne, but somehow also Huygens, regarded sound as a succession of impulses whose precise shape was of little importance to the ear. A simple sound (pure tone) was a succession of similar impulses, while a compound sound contained impulses of different kinds. In particular, there is no mathematically grounded wave theory in which, for example, sounds are represented by superpositions of sinusoidal waves (Euler, 1748). Sound is essentially regarded as a periodic succession of impulses characterized by the value of the period and the intensity.

2. Attempts at Measuring Frequency of Sounds Before Sauveur

Galileo in the *Discorsi* (Galileo, 1638) had highlighted the parameters that influence the frequency of vibration of strings: length, weight per unit length, and tension to which it is subjected. Mersenne discussed the law of the vibrating string at length in his *Harmonie universelle* (Mersenne, 1636). This law is essentially empirical, except for the inverse proportionality between frequency and length, which was somehow proved on a mechanical basis. Expressed in a modern formulation, the law has the form:

$$f = K \frac{1}{l} \sqrt{\frac{F}{\gamma}} \tag{2.1}$$

with the usual meaning of the symbols, where K is a constant of proportionality depending on the units of measure of the parameters appearing in the formula.

However, this law does not give the absolute values of the frequencies, but only the relative ones: for example, one can say that the frequency is halved by doubling the length for the same other parameters, but one cannot say how much the frequency of a string of a given length is worth. Although unable to provide a rigorous demonstration of a formula, Mersenne did have some success in somehow solving the problem of finding the absolute value of frequency. For this purpose, he claimed that it was sufficient to determine experimentally the value of the absolute frequency in a particular case, which was sufficient to determine the constant of proportionality K. He considered the vibrations of a very long string, about 6 m, so that one could easily count its vibrations in a given time. However, he did not insist much on the experiment, both because he was more interested in the method than the result, and because the experiment was not too easy to perform with accuracy, and therefore the result he obtained was not very reliable. The experiment was repeated many years later, in 1860 (Ellis, 1885, p. 298).

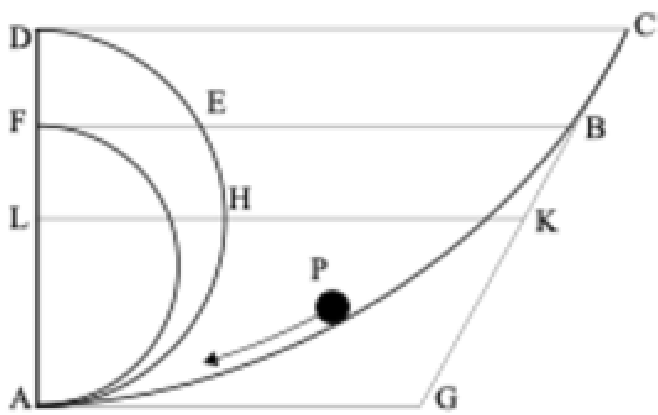

Fig. 1: The motion of a mass point on a cycloid

Another experimental approach to the measurement of absolute frequency was pursued shortly thereafter by Hooke and Huygens with devices broadly classifiable as sirens (Dostrovsky, 1975). Particularly interesting is the attempt of Huygens, probably in 1682, who succeeded in producing a sound of known frequency in unison with the D of a spinet by spinning a series of wheels suitably linked together (Huygens, 1888, vol. 19, p. 375). Somewhat more interesting, though later, is the approach of Vittorio Francesco Stancari (1678-1709). In 1713 a posthumous writing, the memoir *De certa soni mensura costituenda*, reported on an experiment of 1706, not unlike those of Huygens and Hooke, but more refined and, above all, pursued with greater perseverance (Barbieri, 1980, p. 18).

In addition to these experimental attempts, theoretical ones were made during the same period. Huygens had published in 1673 his seminal text *Horologium oscillatorum* in which he demonstrated that, in the absence of friction, the oscillation of a heavy body along a cycloid occurred with a period independent of the amplitude of the oscillation, unlike what happened for a simple pendulum moving on an arc of a circle:

> Proposition XXV In a cycloid with a vertical axis and its vertex at the bottom, the times of descent of a mobile, starting from rest at any point on the curve and reaching the lowest point, are equal to one another and have a ratio to the time of vertical descent along the entire axis of the cycloid equal to that of the semi-circumference of a circle to its diameter. (Huygens, 1673, pp. 184-186).

Note that this proposition allows to obtain the absolute value of the period T of a simple pendulum. With reference to Fig. 1, T is equal to four times the time of descent along PA, and thus four times the time of fall of a heavy body from the vertical AD (the "axe entier" of the cycloid, multiplied by $\frac{\pi}{2}$, and thus:

$$T = 2\pi\sqrt{\frac{2AD}{g}} = 2\pi\sqrt{\frac{l}{g}} \tag{2.2}$$

taking into account that the radius of the osculating circle at the lowest point of the cycloid is 2AD and is equal to the length l of the simple pendulum whose small oscillations occur along the cycloid.

In a fragment published in the *Oeuvres complétes*, Huygens proved that the gravity of a body – in modern terms, the force acting on the body moving in a tangential direction on the cycloid – varies in proportion to the distance from the lowest point of the cycloid (Huygens, 1888, vol. 18, p. 489). This fact, even with the laws of mechanics of the day, enabled him to state that any motion subject to a force proportional to the displacement – as is the case with the vibrating string – is isochronous and harmonic.

In his attempt to find a law for vibrating strings, Huygens proceeded in stages. First, he considered vibrations of strings such as those in Fig. 2, in which a concentrated mass at the midpoint is the object of a force due to the tension of the string that varies linearly with the displacement of the mass. For the string of Fig. 2b, Huygens arrived at the formula:

$$f = \frac{1}{\pi}\sqrt{\frac{g}{2a}}\sqrt{\frac{K}{G}} \tag{2.3}$$

where a is half the length l of the string, K is the value of the weight tending the string, G is the weight of the body G.

Compare the previous relation with the modern frequency formula:

$$f = \frac{1}{2l}\sqrt{\frac{gF}{\gamma}} \tag{2.4}$$

where F is the tension (force of the string) and γ is its mass per unit length, just take $K = K$, $\frac{G}{l} = \gamma$ and you get:

$$f = \frac{1}{\pi l}\sqrt{\frac{gF}{\gamma}} \tag{2.5}$$

The difference between the last two formulas is only the numerical coefficient π instead of 2; therefore, by concentrating the mass in the middle, we get an error of about 50%. Huygens also showed that the transverse oscillations of the string are isochronous with the circular oscillations of the 'compound' pendulum SGH around the axis SK. The modern reader can easily justify this statement by thinking that in

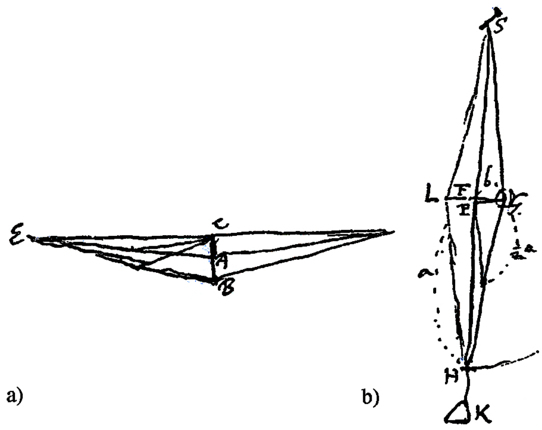

Fig. 2: Vibrations of a string with a concentrated mass

the first case there is a pulling force proportional to the transverse displacement FG due to string tension, in the second case there is a centrifugal force proportional to the distance FG of the oscillation axis SK.

Huygens then considered the case of a mass distributed along the string, or a weightless string loaded by many small weights. But he hardly mentioned the solution. The editor of the *Oeuvres complétes* developed Huygens' suggestion in his own way, following an energy approach, and arrived at the relation (Huygens, 1888, vol. 18, p. 495):

$$f = \frac{\sqrt{10}}{\pi}\frac{1}{2l}\sqrt{\frac{gF}{\gamma}} \tag{2.6}$$

which differs by about 2‰ (excess from the solution believed to be correct).

3. Sauveur First Method physics

Sauveur had already discussed his method for determining the fixed sound at the Académie des sciences of Paris, as reflected in Fontenelle's account in the *Histoire* of 1700 and the minutes of the procés-verbaux of the same year. The procedure was based on counting the beats produced by the sounds of two organ pipes tuned to a minor semitone, i.e. with a ratio of frequencies of $\frac{25}{24}$ and measuring the time interval between each beat; in this way he was able to sound the fixed sound, which for him had to be 100 vibrations per second (i.e., 100 Hz).

In this 1701 memoir, he was a bit more sophisticated; however, it is unclear whether his was simply a more complete exposition or a refinement of the previous procedure. Here is what he wrote:

> Use several organ pipes, which, being open, must be at least two feet long,[1] and tune the sounds of these pipes to the following diatonic intervals, which are so precise that the ear does not perceive the slightest beat in these sounds. Tune 1^0 the octave PA-pa (Ut-ut, 2^0 the fifth PA-BOr (Ut-Sol), or rather PA-BO*ra*; for in order to express the intervals of these sounds with due precision, it is necessary to make use of the decamerides. 3^0. The major third PA-GA*na* (Ut-Mi). 4^0. The minor third PA-go*se* (Ut-Mib). After carefully checking these intervals by comparing these sounds with each other, the just interval of the minor semitone can be obtained accurately go*se*-GA*na*, [the difference between major and minor thirds], the ratio of whose vibrations is 24 and 25. If you play these two pipes, you will hear a beat in their sound at every twenty-fifth vibration of the higher GA*na* (Mi)). (Sauveur, 1701, pp. 360-361)

1 A two-feet organ pipe has a frequency of about 250 Hz.

The first part of the procedure shows how to get two sounds that are a minor semitone apart $\frac{25}{24}$. We start with five different organ pipes that are to be tuned to each other; at the end of the tuning process they would produce the following sounds:

- Pipe A: Note Ut
- Pipe B: Note Ut, one octave higher
- Pipe C: note Sol
- Pipe D: note Mi
- Pipe E: Note Mib

In principle it would have been sufficient to take only three pipes, A, D and E, because A-D form a major third (5:4) and A-E a minor third (6:5), so D-E form a minor semitone (25:24), which is what we wanted to find. The other pipes have only an instrumental function and are useful for achieving correct pitches by checking the correctness of their relative intervals; for example, pipes C and D should produce a minor third.

The second part of the procedure concerns the actual determination of the fixed sound: "After having found the interval between a proposed sound and the fixed sound, it remains to find the fixed sound itself" (Sauveur, 1700, f. 114).

If the pipes D and E are tuned so that they are a semitone minor apart, we can say that when we hear one beat, it means that the sharpest pipe has made 25 vibrations – and the less sharp pipe 24 – and when we hear 4 vibrations, the sharp pipe has made 100 vibrations. But we do not know how long these vibrations last; in particular, we cannot say that they are made in one second. To determine how long it takes the more acute pipe to make 100 vibrations, it is enough to measure how long it takes to have four beats. This will give you the frequency of the acute pipe. It will generally be different from 100 Hz, because there is no method yet to construct a tube that vibrates 100 times per second, and this is precisely the problem that Sauveur set out to solve.

The time between beats is measured with a pendulum by adjusting its length so that four beats are heard at each oscillation. If the time is one second, it means that the acute pipe makes a sound of 100 Hz; if the time is less than one second, the sound is higher and if it is greater, the sound is lower. To find the actual frequency of the acute pipe, simply divide 100 by the time in seconds required for four beats.

However, Sauveur was not satisfied with having found the absolute frequency of an organ pipe; he wanted to be able to emit exactly the fixed sound of 100 Hz. To do this, he used a series of vibrating strings tuned in octaves. Among these strings, he chose the one that contained the sound of the acute pipe in its octave.

At this point Sauveur introduced a movable bridge on the chosen string and placed it so that the string made exactly the sound of the pipe. To obtain the 100 Hz sound, it is enough to move the bridge by the number of merides (43 merides span an octave) obtained with the echometer. (something like a logarithmic scale).

The procedure can only work if the acute pipe emits a sound with a frequency not too far from 100 Hz. It also requires sensitive and trained ears and a very quiet environment, as Sauveur pointed out, for whom to be sure that you have found the right value for the fixed sound:

1. Have a fine ear that can accurately judge the just diatonic intervals.
2. These intervals must be checked in every possible way, and the pipes must be played with the same wind.
3. When comparing the beats with pendulum vibrations, to be sure they are isochronous, we must compare these beats with more than 100 pendulum vibrations.

We do not know how Sauveur arrived at pipe D as a candidate for emitting the fixed sound. He probably

Fig. 3: librations and oscillations of a taut string

used the procedure proposed by Mersenne and then made several tests with his method of beats, adjusting the length of the pipe until the procedure gave a number of beats per second close to four.

4. Sauveur Second Method

The second way used by Sauveur to evaluate the absolute frequency basically consists of the analytical determination of the constant K of the Galileo-Mersenne formula. He started from the idea that the frequency, referring to Fig. 3 of the transversal oscillations of the string in the vertical direction, giving rise to curves of type ADB, is the same as that of the oscillation of the compound pendulum formed by the deformed string – considered as a single rigid body - out of the plane with respect to the axis AB.

This idea is contained, though not in a completely explicit form, in some of Huygens' writings not published in Sauveur's time. It is likely that Sauveur was more or less directly aware of these writings, even if he did not mention them anywhere. In any case, there is an original contribution by Sauveur. Indeed, in order to operationalize Huygens' ideas, one must consider a transversal deformation of the string to be taken as the reference configuration for the compound pendulum. Sauveur proposed for such a configuration that of the string deformed under its own weight, which we know today as a catenary, but which for Sauveur was an unspecified curve to be approximated by a parabola or a circle[2].

A partial justification for this choice is offered in the procès-verbaux of 1714, which is summarized here:

> Though the vibrations of a string are very unequal, and those which it makes at the beginning of its movement are much greater than those which it makes towards the end of its movement, yet the sound which it makes is neither higher nor lower by this fact, though they are more or less easy to hear

However this string was first set in motion, its oscillations diminish to such an extent that they finally return the string to the shape which its weight naturally gives it, so that the last oscillations only cause the string to move laterally, as if the sag of this string had become a pendulum, with the arc of the string suspended at the bottom of this sag, the arc nevertheless moving in a plane (as Mr. Huygens says around its support) (Sauveur, 1714, ff. 177v-178r).

As can be seen, Sauveur's argument is not clear. He seems to state, as an experimental circumstance, that for small oscillations the string vibrates out of the plane formed by the string and the brace, rather than moving in that plane, which is challenging to grasp. The subsequent results will fully validate his approach, however.

To calculate the length of the simple pendulum equivalent to the compound pendulum formed by the deformation of the string, Sauveur used the rules laid down by Huygens in his *Horologium oscillatorium*, but he referred to them only indirectly, preferring to quote the relatively recent work, 1703, of Johann Bernoulli (Bernoulli, 1703). He eventually obtained the same equation as that suggested by the editor of Huygens' works (eq. 2.6).

Still in 1713, about the same time as Sauveur, Brook Taylor (1685-1731) published the 'correct' formula for the frequency of vibration of a string and also the shape of the curve representing the transverse

[2] Most likely Sauveur knew the solution to the catenary problem since it appeared in the Acta eruditorum of 1691, due to Leibniz, Huygens, and Johann Bernoulli (Truesdell, 1960, p. 66), but considered its equation too complex to be used in calculations.

displacements, though limited only to the fundamental mode (i.e., higher modes were not recognized. The memoir with which Taylor presented his results is the *De motu nervi tensi* (Taylor, 1713).

5. Conclusions

The measurement of the frequency of sounds became a problem for musicians relatively late, for several reasons: a) it was not known that sound was a periodic phenomenon characterized by a frequency; b) the frequency of notes has very high values, in the order of a hundred Hz, and is therefore impossible to measure directly, for example by observing oscillations; c) there was no pressing need to measure frequency, since the tuning methods used until then were more than sufficient.

However, at the turn of the 17th century, with the expansion of orchestras and the introduction of a wide variety of musical instruments, and with the spread and internationalization of music, there was a need to standardize pitches, i.e. to ensure that a La played in Berlin today would not sound very different from a La played in Paris two years ago. Until then, the pitches of different notes were not the same in different places and in different times because there was no absolute measure. The pitch of a note was determined rather crudely from the characteristics of the human voice, which was not entirely constant, so that a person could have a lower or higher voice, although the variations were not enormous; there were also approximate pitch devices such as organ pipes or whistles. The nature of sound had now been clarified, it was known that it was a periodic phenomenon characterized by a frequency of vibration, and it was also known that the value of the frequency determined the pitch. The problem of the technical difficulty of measurement remained.

The first effective measurements of frequency were made by Sauveur. He proposed two methods: the first was based on the sound phenomenon of beats, which Sauveur helped to understand. It consists of producing two sounds whose ratio can also be determined by ear; the ratio must be small, close to one, in order to produce beats that can be easily measured. By counting the beats per second and knowing that the frequency of the beats is equal to the difference between the frequencies, it is possible to determine the absolute value of the frequency of the sound producing beats. Sauveur looked at two sounds separated by a semitone minor, with a frequency ratio of 25:24, and was able to reproduce a sound of 100 Hz. The second method proposed by Sauveur is to determine analytically the value of the proportionality constant of the Galileo-Mersenne law. Using Huygens' intuition and results on the compound pendulum, Sauveur succeeded in this task, arriving at an expression that gave a value that was essentially correct by modern standards. The present paper provides a justification for Sauveur's approach and demonstrates his formula, which was previously thought to be unfounded. This demonstration is based on a study of the minutes of the Paris Académie des sciences, which contain arguments that Sauveur did not consider rigorous enough to be published, but which provide insight into his thinking.

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